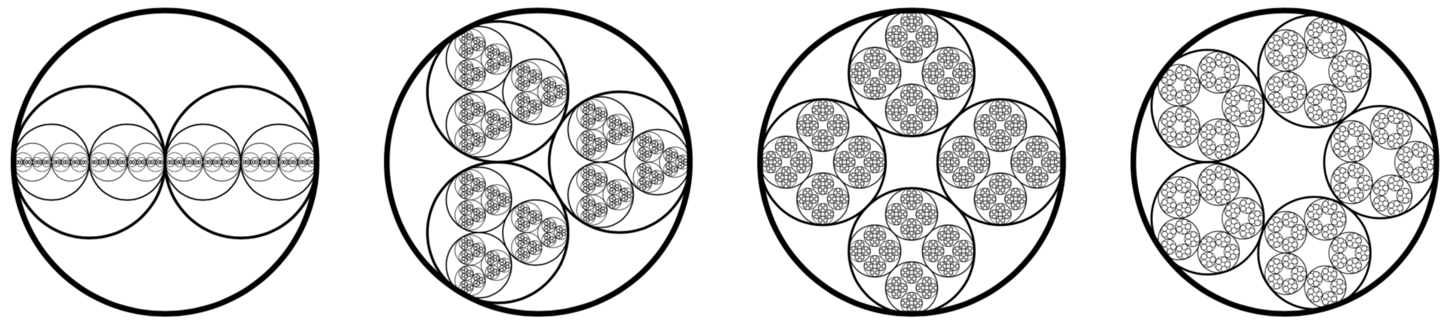
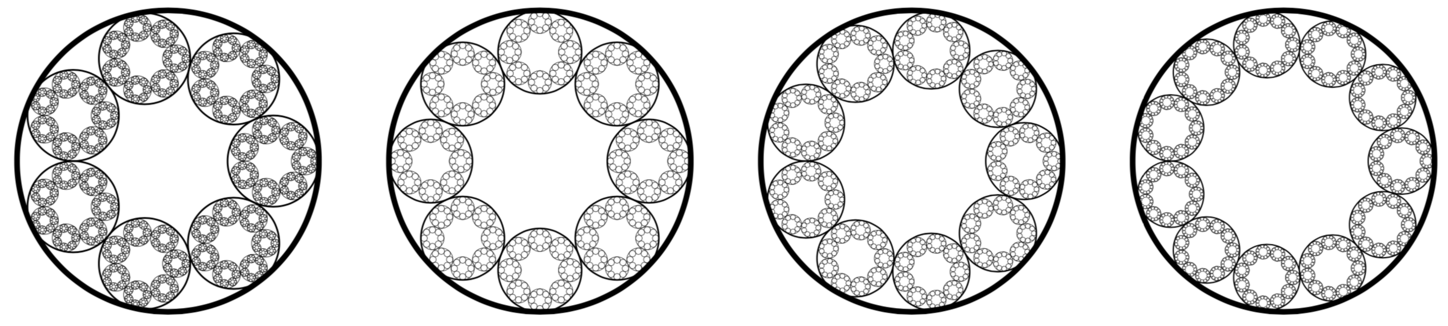


A $\mathfrak{p}$-adic Integral by Combinatorics



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Introduction

Fix an integer $N \geq 2$, an algebraic number field k, and a prime ideal $\mathfrak{p}$ in the ring of integers $\mathcal{O}_k$. Let $q := [\mathcal{O}_k : \mathfrak{p}]$ and let $|\cdot| : k \to \mathbb{R}_{\geq 0}$ be the non-archimedean absolute value defined by $|x| := q^{-\operatorname{ord}_{\mathfrak{p}}(x)}$. Let K be the completion of k with respect to $|\cdot|$, denote the unit balls in K by

$$\mathfrak{o} = \{x \in K : |x| \leq 1\} \qquad \text{and} \qquad \mathfrak{m} = \{x \in K : |x| < 1\},$$

and let μ be the additive Haar measure on K which satisfies $\mu(\mathfrak{o}) = 1$. Writing $(\mathfrak{m})^N$ for the N-fold Cartesian product of $\mathfrak{m}$ and $d\vec{x}$ for Lebesgue integration against the N-fold product measure, define

$$F_N(\vec{\alpha}, \vec{\beta}) := \int_{(\mathfrak{m})^N} (\max_{i<j} |x_i - x_j|)^{\alpha_1} (\min_{i<j} |x_i - x_j|)^{\alpha_2} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}$$

for all suitable $\alpha_i, \beta_{ij} \in \mathbb{C}$. Our main result is that F_N is a finite sum of meromorphic functions in α_i and β_{ij}, each determined explicitly by a finite tree called a *splitting sequence*.

F_N as an Igusa local zeta function

Given a compactly supported locally constant function $\Phi : K^N \to \mathbb{C}$, a continuous homomorphism $w : K^\times \to \mathbb{C}^\times$, and a nonzero polynomial $f(\vec{x}) \in K[\vec{x}] = K[x_1, \ldots, x_N]$, the associated *Igusa local zeta function* is defined by

$$Z_\Phi(w, K, f) := \int_{K^N} \Phi(\vec{x}) w(f(\vec{x})) \, d\vec{x}.$$

Explicit values of $Z_\Phi(w, K, f)$ for certain Φ, w, and f are computed in [Igu89] and many general properties of Z_Φ are summarized in [Den91]. In particular, Denef showed by decomposing K^N that $Z_\Phi(w, K, f)$ is a rational function in q^s where s is the exponent of w. We find new values of $Z_\Phi(w, K, f)$ by noting that $Z_\Phi(w, K, f) = F_N(\vec{\alpha}, \vec{\beta})$ when

$$\begin{aligned} \Phi(\vec{x}) &= \mathbf{1}_{(\mathfrak{m})^N}(\vec{x}) (\max_{i<j} |x_i - x_j|)^{\alpha_1} (\min_{i<j} |x_i - x_j|)^{\alpha_2}, && \Re(\alpha_i) \geq 0, \\ w(y) &= |y|^s, && \Re(s) \geq 0, \\ f(\vec{x}) &= \prod_{i<j} (x_i - x_j)^{d_{ij}}, && d_{ij} \in \mathbb{Z}_{\geq 0}, \end{aligned}$$

and $\beta_{ij} = d_{ij} s$, and we will see that this $Z_\Phi(w, K, f)$ is rational in all q^{α_i} and $q^{\beta_{ij}}$.

F_N as the partition function for a log gas in K

In the classical real setting, a *one-dimensional log gas* is a system of N charged particles distributed along the real line, subject to a repulsive logarithmic Coulomb potential and an attractive harmonic potential at the origin as in [For10]. The particle locations $x_i \in \mathbb{R}$ are organized into a vector $\vec{x} = (x_1, \ldots, x_N)^t \in \mathbb{R}^N$ called a *microstate*, to which we associate an *energy*

$$V(\vec{x}) := \sum_{i=1}^{N} \frac{(q_i x_i)^2}{2\beta} - \sum_{i<j} q_i q_j \log |x_i - x_j|_\infty$$

where $|\cdot|_\infty$ denotes the usual (archimedean) absolute value on $\mathbb{R}$, the parameter $\beta \geq 0$ is the *inverse temperature* (or "coldness") of the system, and q_i is the charge of the particle at x_i. The *canonical partition function* for this system is

$$Z_N(\beta) := \int_{\mathbb{R}^N} e^{-\beta V(\vec{x})} \, d\vec{x} = \int_{\mathbb{R}^N} \prod_{i=1}^{N} e^{-\frac{1}{2}(q_i x_i)^2} \prod_{i<j} |x_i - x_j|_\infty^{q_i q_j \beta} d\vec{x},$$

and $\frac{1}{Z_N(\beta)} e^{-\beta V(\vec{x})}$ is a probability density on $\mathbb{R}^N$ from which many physical qualities of the system are derived. In the special case that $q_i = 1$ for all i we recognize $Z_N(\beta)$ as the *Mehta integral*, which extends to the meromorphic function given by

$$Z_N(\beta) = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}.$$

The interesting history and proof of this formula can be found respectively in [FW08] and [For10]. Now consider a nonarchimedean analogue, where the N particles have locations $x_i \in K$ and charges q_i and are confined to $\mathfrak{m}$ by an infinite potential outside $\mathfrak{m}$. The energy associated to a microstate $\vec{x} = (x_1, \ldots, x_N)^t \in K^N$ in this system is

$$V(\vec{x}) := \begin{cases} -\sum_{i<j} q_i q_j \log |x_i - x_j| & \text{if } x_i \in \mathfrak{m} \text{ for all } i, \\ \infty & \text{otherwise,} \end{cases}$$

and so the canonical partition function is

$$Z_N(\beta) := \int_{K^N} e^{-\beta V(\vec{x})} \, d\vec{x} = \int_{K^N} \mathbf{1}_{(\mathfrak{m})^N}(\vec{x}) \prod_{i<j} |x_i - x_j|_v^{q_i q_j \beta} \, d\vec{x} = F_N(0, \vec{\beta})$$

where $\beta_{ij} = q_i q_j \beta$ and $\beta \geq 0$ is the coldness of the system. In particular, we have a probability density function $\frac{1}{Z_N(\beta)} \prod_{i<j} |x_i - x_j|^{q_i q_j \beta}$ on the microstates $\vec{x} \in (\mathfrak{m})^N$ and

$$\mathbb{E}\left[(\max_{i<j} |X_i - X_j|_v)^{\alpha_1} (\min_{i<j} |X_i - X_j|)^{\alpha_2}\right] = \frac{F_N(\vec{\alpha}, \vec{\beta})}{F_N(0, \vec{\beta})}.$$

Basic properties of K

The topological and algebraic properties of K work together to give a convenient visualization of $\mathfrak{m}$. We summarize some of them below (see [Wei95] for proofs):

Proposition 1 (Properties of K)**.**

(a) The closed unit ball $\mathfrak{o}$ is a discrete valuation ring with fraction field K, maximal ideal $\mathfrak{m}$, and unit group $\mathfrak{o}^\times = \mathfrak{o} \setminus \mathfrak{m} = \{x \in K : |x| = 1\}$.

(b) The image of $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is $q^{\mathbb{Z}} \cup \{0\}$, so every ball containing 0 in K has the form

$$\mathfrak{m}^n = \{x \in K : |x| \leq q^{-n}\}, \qquad n \in \mathbb{Z}.$$

These $\mathfrak{m}^n$ (not to be confused with $(\mathfrak{m})^n$) comprise the group of fractional ideals of K. Each satisfies $\mathfrak{m}^n \cap k = \mathfrak{p}^n$, and for every $r \geq n$ we have an isomorphism of abelian groups $\mathfrak{p}^n / \mathfrak{p}^r \to \mathfrak{m}^n / \mathfrak{m}^r$.

(c) Every ball in K is open, compact, and of the form $a + \mathfrak{m}^n$ with $a \in K$ and $n \in \mathbb{Z}$. The measure of each ball is given by $\mu(a + \mathfrak{m}^n) = q^{-n}$.

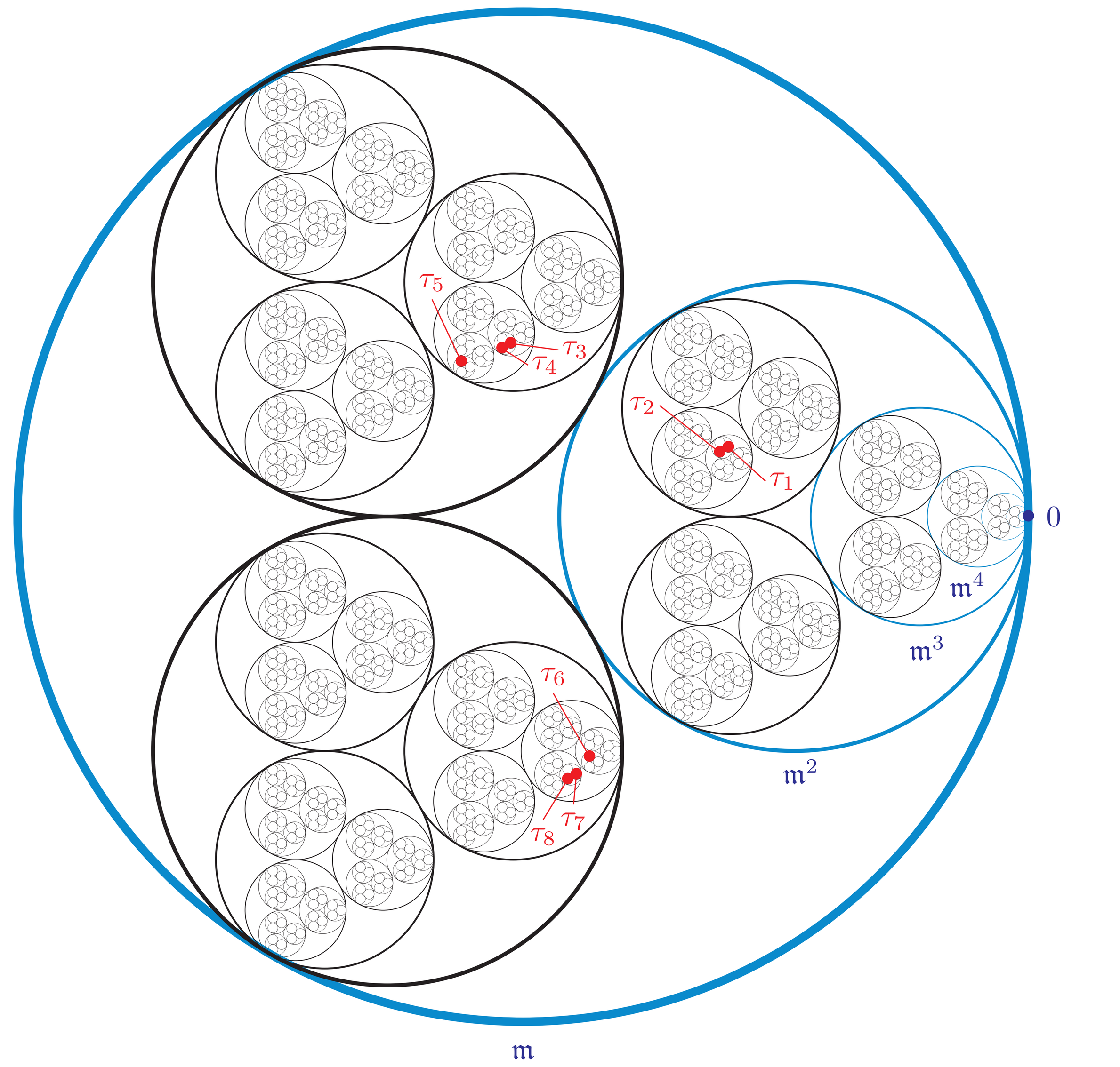


Figure 1: All cosets of $\mathfrak{m}^2, \mathfrak{m}^3, \ldots, \mathfrak{m}^7$ in $\mathfrak{m}$ where $q = 3$. Eight cosets mod $\mathfrak{m}^7$ are labeled in red.

What is a splitting sequence?

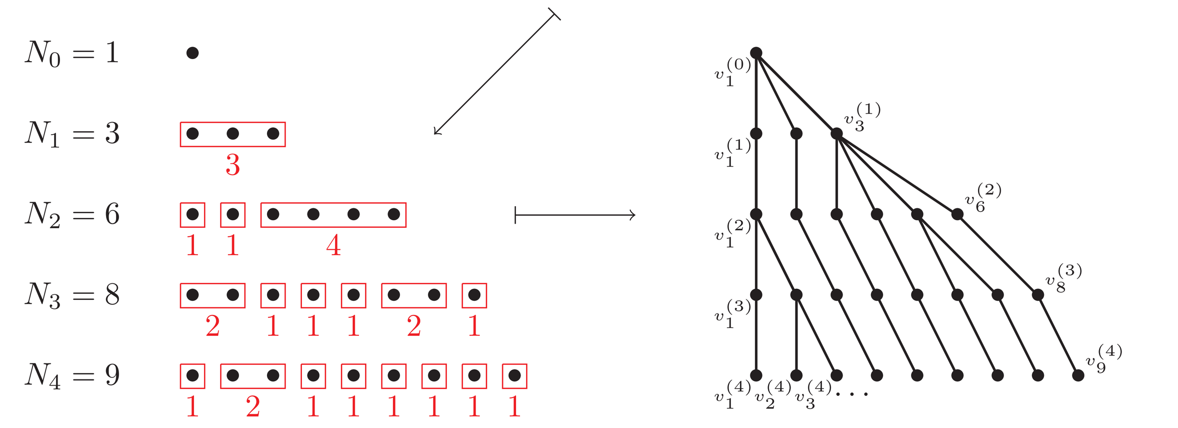
Definition 2. A *splitting sequence of* N is a tuple $\vec{C} = (C_0, C_1, \ldots, C_{L-1})$ of compositions $C_\ell = [\lambda_1^{(\ell)}, \lambda_2^{(\ell)}, \ldots, \lambda_{N_\ell}^{(\ell)}] \vdash N_{\ell+1}$ such that $1 = N_0 < N_1 < \cdots < N_{L-1} < N_L = N$. In this case we write $\vec{C} \pitchfork N$ and call $L(\vec{C}) := L$ the *length* of $\vec{C}$.

The key property of splitting sequences is the correspondence

$$\left\{\vec{C} \pitchfork N\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix} \textit{ordered rooted finite trees with splitting} \\ \textit{in all depths and } N \textit{ leaves at the bottom} \end{matrix}\right\}.$$

Figure 2: A two-step construction of the tree corresponding to a splitting sequence of 9.

$$\vec{C} = ([3], [1,1,4], [2,1,1,1,2,1], [1,2,1,1,1,1,1,1])$$



Definition 3. Let $\vec{C} \pitchfork N$, $n \in \mathbb{N}$, and $\alpha_i, \beta_{ij} \in \mathbb{C}$. For $\ell \in \{0, 1, \ldots, L(\vec{C}) - 1\}$ define

$$\Lambda_m^{(\ell)} := \{i : v_i^{(L(\vec{C}))} \text{ is a descendant of } v_m^{(\ell)} \text{ in the tree for } \vec{C}\} \quad \text{for } m \in \{1, 2, \ldots, N_\ell\},$$

$$M_\ell(\vec{C}; n) := \prod_{m=1}^{N_\ell} \frac{1}{n} \binom{n}{\lambda_m^{(\ell)}}, \quad \text{and} \quad E_\ell(\vec{\alpha}, \vec{\beta}, \vec{C}) := \delta_{0\ell} \alpha_1 + \alpha_2 + \sum_{m=1}^{N_\ell} \sum_{\substack{i,j \in \Lambda_m^{(\ell)} \\ i<j}} \left(\beta_{ij} + \frac{2}{|\Lambda_m^{(\ell)}|}\right).$$

The main result

Theorem 4 (Main Theorem)**.** Define the open sets

$$\Omega_+ := \left\{(\vec{\alpha}, \vec{\beta}) \in \mathbb{C}^2 \times \mathbb{C}^{\binom{N}{2}} : \Re(E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{C})) > 0 \text{ for all } \vec{C} \pitchfork N, \text{ all } \sigma \in S_N, \text{ and all } \ell\right\}$$

and

$$\Omega := \left\{(\vec{\alpha}, \vec{\beta}) \in \mathbb{C}^2 \times \mathbb{C}^{\binom{N}{2}} : E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{C}) \notin \tfrac{2\pi i \mathbb{Z}}{\log(q)} \text{ for all } \vec{C} \pitchfork N, \text{ all } \sigma \in S_N, \text{ and all } \ell\right\}$$

where $\sigma(\vec{\beta}) = (\beta_{\sigma^{-1}(i)\sigma^{-1}(j)})$. The function F_N defined by

$$F_N(\vec{\alpha}, \vec{\beta}) := \int_{(\mathfrak{m})^N} (\max_{i<j} |x_i - x_j|)^{\alpha_1} (\min_{i<j} |x_i - x_j|)^{\alpha_2} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}$$

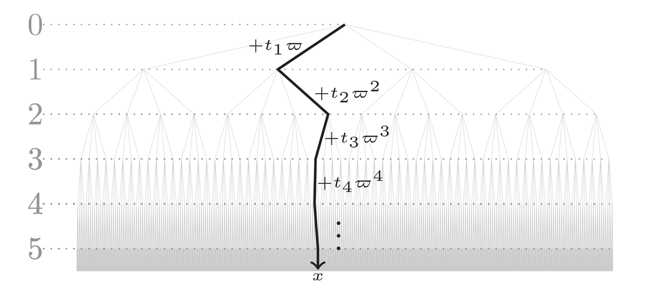
is analytic on Ω_+ and extends to the analytic function on Ω defined by

$$F_N(\vec{\alpha}, \vec{\beta}) = \frac{1}{q^N} \sum_{\sigma \in S_N} \sum_{\vec{C} \pitchfork N} \prod_{\ell=0}^{L(\vec{C})-1} \frac{M_\ell(\vec{C}; q)}{q^{E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{C})} - 1}.$$

A proof outline

(1) **[Totally order $\mathfrak{m}$]** Fix an element $\varpi \in \mathfrak{p} \setminus \mathfrak{p}^2$ and a full set of representatives T for the cosets of $\mathfrak{p} \subset \mathcal{O}_k$, and recall $|T| = |\mathcal{O}_k / \mathfrak{p}| = q$. It is well known that each $x \in \mathfrak{m}$ can be written uniquely in the form

$$x = \sum_{n=1}^{\infty} t_n \varpi^n, \quad t_n \in T.$$



Now fix a total order $\prec$ on T such that 0 is the least element. By identifying each $x \in \mathfrak{m}$ with its coefficient word $t_1 t_2 t_3 \ldots$, we define a total (lexicographic) order $\prec$ on all of $\mathfrak{m}$ using the "alphabet" $(T, \prec)$. Note that each $x \in \mathfrak{m}$ can be visualized as a path down an ordered q-ary tree, in which $x \prec y$ if and only if the path for x is (eventually) to the left of the path for y.

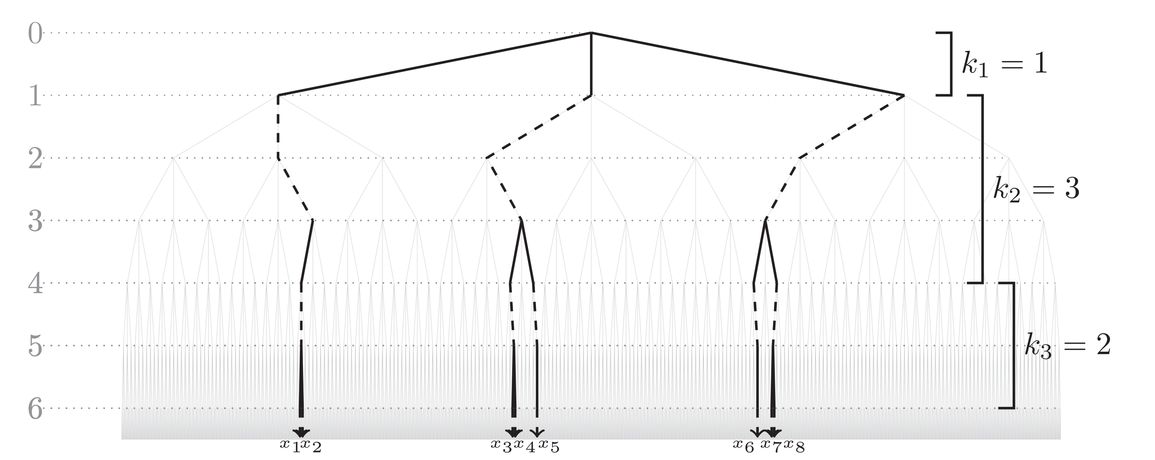
(2) **[Decompose $(\mathfrak{m})^N$ via Weyl chambers]** Define the *fundamental Weyl chamber* in $(\mathfrak{m})^N$ by $\mathcal{W} := \{\vec{x} \in (\mathfrak{m})^N : 0 \prec x_1 \prec x_2 \prec \cdots \prec x_N\}$. Writing $f(\vec{x}, \vec{\alpha}, \vec{\beta})$ for the integrand of F_N, neglecting a set of measure zero, and changing variables gives

$$F_N(\vec{\alpha}, \vec{\beta}) = \int_{(\mathfrak{m})^N} f(\vec{x}, \vec{\alpha}, \vec{\beta}) \, d\vec{x} = \sum_{\sigma \in S_N} \int_{\mathcal{W}} f(\vec{x}, \vec{\alpha}, \sigma(\vec{\beta})) \, d\vec{x}. \qquad (*)$$

(3) **[Decompose $\mathcal{W}$ into level sets of $f(-, \vec{\alpha}, \sigma(\vec{\beta}))$]** Each $\vec{x} \in \mathcal{W}$ determines a unique pair $(\vec{C}, \vec{k})$ where $\vec{C} \pitchfork N$ and $\vec{k} \in \mathbb{N}^{L(\vec{C})}$ (see Figure 3), so $\mathcal{W}$ is a countable disjoint union of the sets $\mathcal{W}(\vec{C}, \vec{k}) := \{\vec{x} \in \mathcal{W} : \vec{x} \text{ determines } (\vec{C}, \vec{k})\}$. In fact, $f(-, \vec{\alpha}, \sigma(\vec{\beta}))$ is constant on each $\mathcal{W}(\vec{C}, \vec{k})$ and $\mu^N(\mathcal{W}(\vec{C}, \vec{k}))$ can be found by counting, which yields

$$\int_{\mathcal{W}(\vec{C}, \vec{k})} f(\vec{x}, \vec{\alpha}, \sigma(\vec{\beta})) \, d\vec{x} = \frac{1}{q^N} \prod_{\ell=0}^{L(\vec{C})-1} M_\ell(\vec{C}, q) q^{-E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{C}) k_{\ell+1}}.$$

The theorem follows by summing over all $\vec{k}$ and $\vec{C}$ and returning to $(*)$.



$$[\Lambda_1^{(0)}] = [\{1,2,3,4,5,6,7,8\}]$$

$$[\Lambda_1^{(1)}, \Lambda_2^{(1)}, \Lambda_3^{(1)}] = [\{1,2\}, \{3,4,5\}, \{6,7,8\}]$$

$$[\Lambda_1^{(2)}, \Lambda_2^{(2)}, \Lambda_3^{(2)}, \Lambda_4^{(2)}, \Lambda_5^{(2)}] = [\{1,2\}, \{3,4\}, \{5\}, \{6\}, \{7,8\}]$$

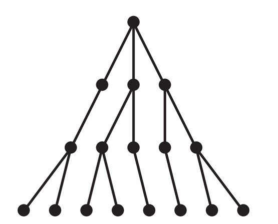


Figure 3: The subgroup $\mathfrak{m}^2 \subset \mathfrak{m}$ in Figure 1 has $q = 3$ distinct cosets, which in counterclockwise order are $\mathfrak{m}^2$, $t\varpi + \mathfrak{m}^2$, and $t'\varpi + \mathfrak{m}^2$ where $T = \{0, t, t'\}$. The ordering $0 \prec t \prec t'$ defines a total order on $\mathfrak{m}$ such that the cartesian product of the red cosets labeled $\tau_1, \tau_2, \ldots, \tau_8$ (in order) from Figure 1 is contained in $\mathcal{W}$. Let $\vec{x} = (x_1, x_2, \ldots, x_8)$ be an element of that product and superimpose the paths for $x_1, x_2, \ldots, x_8$ as shown above. Collapsing the dashed path segments reveals the tree of the splitting sequence $\vec{C} = ([3], [1,2,2], [2,2,1,1,2]) \pitchfork 8$ and measuring their lengths determines $\vec{k} = (1,3,2)$. The sets $\Lambda_m^{(\ell)}$ corresponding to $\vec{C}$ are also tabulated above.

Recursive construction of splitting sequences

For a particular $N \geq 2$, one needs all $\vec{C} \pitchfork N$ explicitly in order to compute F_N using Theorem 4. They can be constructed recursively as follows. Given $\vec{C} \pitchfork N$, let $L = L(\vec{C})$ and construct a family of splitting sequences $\vec{C}' \pitchfork (N+1)$ using two types of modifications:

(1) **[Add a row]** A composition of $(N+1)$ having N parts must be comprised of $(N-1)$ 1's and a single 2. Choosing one such composition $C_L \vdash (N+1)$ and appending it to $\vec{C} = (C_0, C_1, \ldots, C_{L-1})$ yields a splitting sequence $\vec{C}' = (C_0, C_1, \ldots, C_L) \pitchfork (N+1)$.

(2) **[Add a node]** The last composition in $\vec{C}$ has the form $C_{L-1} = [\lambda_1, \lambda_2, \ldots, \lambda_{N_{L-1}}] \vdash N$. We may construct $C'_{L-1} \vdash (N+1)$ from C_{L-1} by increasing one of the parts λ_m by 1, which yields a splitting sequence $\vec{C}' = (C_0, C_1, \ldots, C_{L-2}, C'_{L-1}) \pitchfork (N+1)$.

Each $\vec{C} \pitchfork N$ yields at most $2N - 1$ distinct $\vec{C}' \pitchfork (N+1)$ via (1) and (2). Every $\vec{C}' \pitchfork (N+1)$ can be constructed from some $\vec{C} \pitchfork N$ in this way, so induction on N then gives:

Proposition 5. For $N \geq 2$ we have $\#\{\vec{C} \pitchfork N\} \leq (2N-3)!!$ with equality only if $N = 2, 3$.

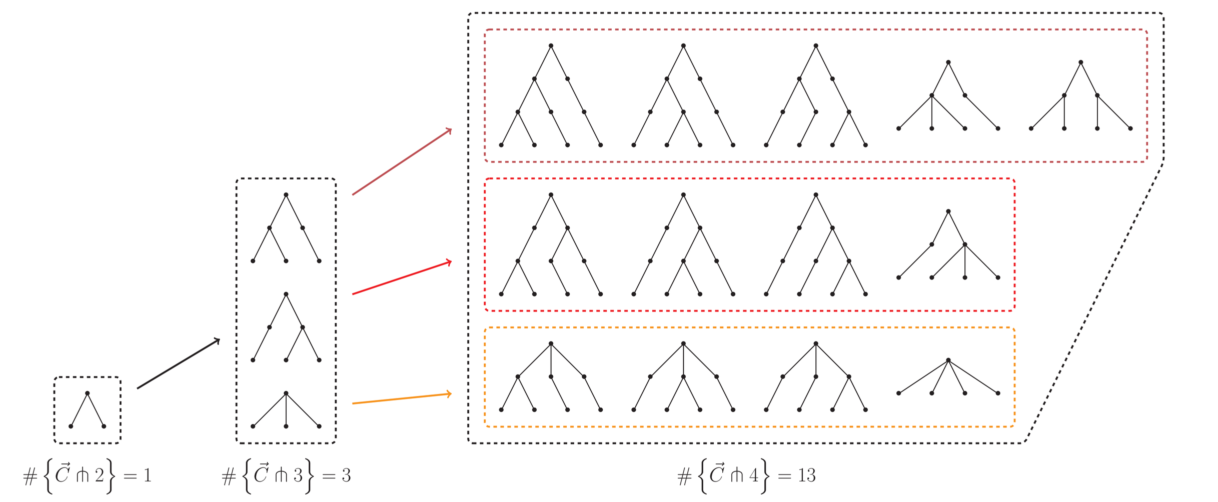


Figure 4: All splitting sequence trees for $N = 2, 3, 4$. The splitting sequences of 4 are organized into three groups constructed respectively from the three splitting sequences of 3.

Example: $K = \mathbb{Q}_p$ with $\alpha_1 = \alpha_2 = 0$ and $\beta_{ij} = \beta$

Recall $\mathfrak{m} = p\mathbb{Z}_p$ and $q = p$ is prime. Given $\vec{C} \pitchfork N$, Definition 3 implies $M_\ell(\vec{C}; p) \in \mathbb{N}$ if $\lambda_m^{(\ell)} \leq p$ for all m and otherwise $M_\ell(\vec{C}; p) = 0$. Moreover, since $\alpha_1 = \alpha_2 = 0$ and $\beta_{ij} = \beta$ for all ij, Definition 3 gives

$$E_\ell(0, \sigma(\vec{\beta}), \vec{C}) = \sum_{m=1}^{N_\ell} \sum_{\substack{i,j \in \Lambda_m^{(\ell)} \\ i<j}} \left(\beta + \frac{2}{|\Lambda_m^{(\ell)}|}\right) = \Gamma_\ell \beta + N - N_\ell, \quad \text{where} \quad \Gamma_\ell := \sum_{m=1}^{N_\ell} \binom{|\Lambda_m^{(\ell)}|}{2},$$

for each ℓ and σ. Using the definitions of Ω and Ω_+ in Theorem 4, it is easily verified that, if $(0, 0, \beta, \beta, \ldots, \beta) = (\vec{\alpha}, \vec{\beta}) \in \Omega$ then β is not contained in

$$P := \bigcup_{\vec{C} \pitchfork N} \left\{-\frac{N - N_\ell}{\Gamma_\ell} + \frac{2\pi i n}{\Gamma_\ell \log(p)} : n \in \mathbb{Z} \text{ and } \ell \in \{0, 1, \ldots, L(\vec{C}) - 1\}\right\},$$

and $(0, 0, \beta, \beta, \ldots, \beta) = (\vec{\alpha}, \vec{\beta}) \in \Omega_+$ if and only if $\Re(\beta) > -\frac{2}{N}$. Now Theorem 4 implies the following:

Corollary 6. The holomorphic function Z_N defined by

$$Z_N(\beta) = \int_{(p\mathbb{Z}_p)^N} \prod_{i<j} |x_i - x_j|_v^\beta \, d\vec{x} \qquad \text{on} \qquad \{\beta \in \mathbb{C} : \Re(\beta) > -2/N\}$$

has an analytic continuation to $\mathbb{C} \setminus P$ given by

$$Z_N(\beta) = \frac{N!}{p^N} \sum_{\vec{C} \pitchfork N} \prod_{\ell=0}^{L(\vec{C})-1} \frac{M_\ell(\vec{C}; p)}{p^{\Gamma_\ell \beta + N - N_\ell} - 1}.$$

This formula is an analog of Mehta's integral, which for $\beta \geq 0$ gives an explicit form of the partition function for a log gas of N unit charges in $p\mathbb{Z}_p$. For example, if $N = 4$ we use the set of splitting sequences $\{\vec{C} \pitchfork 4\}$ in Figure 4 to compute

$$\begin{aligned} Z_4(\beta) = \frac{4!}{p^4} \Bigg(& \frac{\frac{1}{p}\binom{p}{4}}{(p^{6\beta+3} - 1)} + \frac{\frac{2}{p^2}\binom{p}{2}\binom{p}{3}}{(p^{6\beta+3} - 1)(p^{3\beta+2} - 1)} + \frac{\frac{4}{p^3}\binom{p}{2}^3}{(p^{6\beta+3} - 1)(p^{3\beta+2} - 1)(p^{\beta+1} - 1)} \\ & + \frac{\frac{3}{p^2}\binom{p}{3}\binom{p}{2}}{(p^{6\beta+3} - 1)(p^{\beta+1} - 1)} + \frac{\frac{1}{p^3}\binom{p}{2}^3}{(p^{6\beta+3} - 1)(p^{2\beta+2} - 1)} + \frac{\frac{2}{p^3}\binom{p}{2}^3}{(p^{6\beta+3} - 1)(p^{2\beta+2} - 1)(p^{\beta+1} - 1)} \Bigg). \end{aligned}$$

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