log-Coulomb gas in a nonarchimedean local field

Joe Webster

University of Oregon

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The known story for $\mathbb{R}$

- A one-dimensional **log-Coulomb gas** (or **log-gas**) is a system of $N \geq 2$ point charges $q_1, q_2, \ldots, q_N > 0$ constrained to a line and subject to a "repulsive log-Coulomb potential."

- Let $x_i \in \mathbb{R}$ be the location of q_i and call $\vec{x} = (x_1, x_2, \ldots, x_N)$ a **microstate**. Let $T > 0$ be a fixed temperature.

- Define the potential energy of a microstate $\vec{x} \in \mathbb{R}^N$ by

$$V(\vec{x}) := \frac{1}{2} T \|\vec{x}\|^2 - \sum_{i<j} q_i q_j \log |x_i - x_j| \ .$$

- *Rough interpretation:* A microstate has high energy if charges are far from the origin (the quadratic term) or close together (the logarithmic term).

The known story for $\mathbb{R}$ (continued)

- Call $\beta = \frac{1}{T}$ the "coldness" of the system and set $\beta_{ij} := q_i q_j \beta$ for all $i < j$.

- Define the **canonical partition function** Z_N by

$$Z_N(\beta) := \int_{\mathbb{R}^N} e^{-\beta V(\vec{x})}\, d\vec{x} = \int_{\mathbb{R}^N} e^{-\frac{1}{2}\|\vec{x}\|^2} \prod_{i<j} |x_i - x_j|^{\beta_{ij}}\, d\vec{x} \; .$$

- *Fundamental idea of Boltzmann statistics:* The microstates $\vec{x} \in \mathbb{R}^N$ have probability density $\frac{1}{Z_N(\beta)} e^{-\beta V(\vec{x})}$.

- *Rough interpretation:* High energy microstates are less probable. This effect is more severe if the system is cold ($\beta \gg 0$) and less severe if the system is hot ($\beta \approx 0$).

The known story for $\mathbb{R}$ (continued)

- It is hard to compute $Z_N(\beta)$ for general q_i. In the special case $q_i = 1$ for all i, we have $\beta_{ij} = \beta$ for all ij and $Z_N(\beta)$ becomes the value known as **Mehta's integral**:

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\|\vec{x}\|^2} \prod_{i<j} |x_i - x_j|^\beta \, d\vec{x} = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)} \, .$$

- *Early 1960's:* Mehta and Dyson proved the above formula only for $\beta = 1, 2, 4$, while developing random matrix theory.

- *Late 1970's:* A lucky encounter with Selberg's integral led Bombieri to a clever proof. It is valid for all complex β at which the integral converges.

Motivating questions

- How do things change if we replace $\mathbb{R}$ with a *non*-archimedean local field, such as $\mathbb{Q}_p$? What becomes easier/harder?
- Are there interesting properties common to p-adic log-gases and real log-gas? Is there a unified way to handle them all?
- What do these "local log-gases" together imply about adèle rings and idèle groups of number fields?
- *Today's goal:* Discuss the first question and some examples.

Basic setup of a nonarchimedean local field

- Let K be a non-archimedean local field with discrete valuation $v : K \to \mathbb{Z} \cup \{\infty\}$. Recall $v(x+y) \geq \min\{v(x), v(y)\}$ with $v(0) = \infty$, and v is a homomorphism of $(K^\times, \cdot)$ onto $(\mathbb{Z}, +)$.
- $\mathfrak{o} := \{x \in K : v(x) \geq 0\}$ is a DVR, $\mathfrak{m} := \{x \in K : v(x) > 0\}$ is its maximal ideal, and local compactness of K provides a unique integer $q = p^f$ satisfying $\mathfrak{o}/\mathfrak{m} \cong \mathbb{F}_q$.
- Note K is one of the following:
  - a finite extension of $\mathbb{Q}_p$ with v a rescaled extension of ord_p
  - the field $\mathbb{F}_q((t))$ of formal Laurent series with $v = \mathrm{ord}_t$
- Pick the absolute value $|\cdot|$ on K satisfying $|x| = q^{-v(x)}$, note

$$\mathfrak{o} = \{x \in K : |x| \leq 1\} \qquad \text{and} \qquad \mathfrak{m} = \{x \in K : |x| < 1\} .$$

- Pick the additive Haar measure μ satisfying $\mu(\mathfrak{o}) = 1$.

log-gas in K

- Suppose $q_1, q_2, \ldots, q_N > 0$, $\beta > 0$, and $\beta_{ij} = q_i q_j \beta$ as before. Now suppose q_i is located at $x_i \in K$ and define the potential energy of a microstate $\vec{x} \in K^N$ by

$$V(\vec{x}) := \begin{cases} -\sum_{i<j} q_i q_j \log |x_i - x_j| & \text{if all } x_i \in \mathfrak{o}, \\ \infty & \text{otherwise.} \end{cases}$$

- Set $d\vec{x} := d\mu^N(\vec{x})$ and define the canonical partition function:

$$Z_N(\beta) := \int_{K^N} e^{-\beta V(\vec{x})} \, d\vec{x} = \int_{\mathfrak{o}^N} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x} \; .$$

- The microstates $\vec{x} \in K^N$ have probability density

$$\frac{1}{Z_N(\beta)} e^{-\beta V(\vec{x})} = \frac{1}{Z_N(\beta)} \mathbf{1}_{\mathfrak{o}^N}(\vec{x}) \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \; .$$

A preview of $Z_N(\beta)$ values

- $Z_2(\beta) = \frac{(q-1)q^{\beta_{12}}}{q^{\beta_{12}+1}-1}$
- $Z_3(\beta) = \frac{(q-1)q^{\beta_{12}+\beta_{13}+\beta_{23}}}{q^{\beta_{12}+\beta_{13}+\beta_{23}+2}-1} \cdot \left[(q-2) + \frac{q-1}{(q^{\beta_{12}+1}-1)} + \frac{q-1}{(q^{\beta_{13}+1}-1)} + \frac{q-1}{(q^{\beta_{23}+1}-1)}\right]$
- $$\begin{aligned} Z_4(\beta) = & \frac{(q-1)q^{\beta_{12}+\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}+\beta_{34}}}{q^{\beta_{12}+\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}+\beta_{34}+3}-1} \cdot \Big\{(q-2)(q-3) \\ & + (q-2)\left[\frac{q-1}{q^{\beta_{12}+1}-1} + \frac{q-1}{q^{\beta_{23}+1}-1} + \frac{q-1}{q^{\beta_{13}+1}-1} + \frac{q-1}{q^{\beta_{14}+1}-1} + \frac{q-1}{q^{\beta_{24}+1}-1} + \frac{q-1}{q^{\beta_{34}+1}-1}\right] \\ & + \frac{q-1}{q^{\beta_{12}+\beta_{23}+\beta_{13}+2}-1}\left[(q-2) + \frac{q-1}{q^{\beta_{12}+1}-1} + \frac{q-1}{q^{\beta_{23}+1}-1} + \frac{q-1}{q^{\beta_{13}+1}-1}\right] \\ & + \frac{q-1}{q^{\beta_{12}+\beta_{14}+\beta_{24}+2}-1}\left[(q-2) + \frac{q-1}{q^{\beta_{12}+1}-1} + \frac{q-1}{q^{\beta_{14}+1}-1} + \frac{q-1}{q^{\beta_{24}+1}-1}\right] \\ & + \frac{q-1}{q^{\beta_{13}+\beta_{14}+\beta_{34}+2}-1}\left[(q-2) + \frac{q-1}{q^{\beta_{13}+1}-1} + \frac{q-1}{q^{\beta_{14}+1}-1} + \frac{q-1}{q^{\beta_{34}+1}-1}\right] \\ & + \frac{q-1}{q^{\beta_{23}+\beta_{34}+\beta_{24}+2}-1}\left[(q-2) + \frac{q-1}{q^{\beta_{23}+1}-1} + \frac{q-1}{q^{\beta_{34}+1}-1} + \frac{q-1}{q^{\beta_{24}+1}-1}\right] \\ & + \frac{q-1}{q^{\beta_{12}+1}-1} \cdot \frac{q-1}{q^{\beta_{34}+1}-1} + \frac{q-1}{q^{\beta_{13}+1}-1} \cdot \frac{q-1}{q^{\beta_{24}+1}-1} + \frac{q-1}{q^{\beta_{14}+1}-1} \cdot \frac{q-1}{q^{\beta_{23}+1}-1}\Big\} \end{aligned}$$

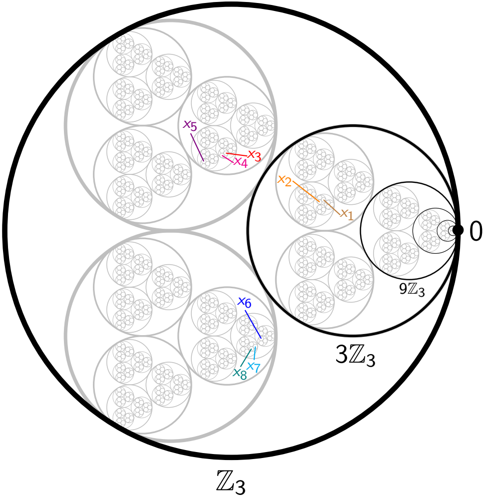
Average values

- The **expectation** (or **average value**) of a Borel measurable function $f : \mathfrak{o}^N \to \mathbb{C}$ is

$$\mathbb{E}[f(\vec{X})] := \frac{1}{Z_N(\beta)} \int_{\mathfrak{o}^N} f(\vec{x}) \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x} \, .$$

- Some meaningful examples of $f(\vec{x})$:
  - $\prod_i |x_i|^{s_i}$ = an unramified quasicharacter of $(K^\times)^N$ ($s_i \in \mathbb{C}$)
  - $\min_{i<j} v(x_i - x_j) = \min\{n : x_i \not\equiv x_j \bmod \mathfrak{m}^{n+1} \text{ for some } i < j\}$
  - $\max_{i<j} v(x_i - x_j) = \min\{n : x_i \not\equiv x_j \bmod \mathfrak{m}^{n+1} \text{ for all } i < j\}$
  - $V(\vec{x})$ = the total potential energy of the system
  - $\min_{i<j} |x_i - x_j|$ = the minimum distance between charges
  - $\max_{i<j} |x_i - x_j|$ = the diameter of the gas

Example: eight unit charges in $\mathbb{Z}_3$



A microstate $\vec{x} \in \mathbb{Z}_3^8$ with...

- $\prod_i |x_i|^{s_i} = 3^{-(s_1+s_2)}$
- $\min_{i<j} v(x_i - x_j) = 0$
- $\max_{i<j} v(x_i - x_j) = 5$
- $V(\vec{x}) = 27\log(3)$
- $\min_{i<j} |x_i - x_j| = 3^{-5}$
- $\max_{i<j} |x_i - x_j| = 1$

Expectation of quasicharacters and energy

- *Unramified quasi-characters:* Given $\vec{\beta} = (\beta_{ij})$ as before and $(s_1, s_2, \ldots, s_N) \in \mathbb{C}^N$ with $\Re(s_i) > -1$, set $\beta_{i(N+1)} := s_i$ and

$$Z^*_{N+1}(\beta) := \int_{\mathfrak{o}^{N+1}} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}.$$

  A simple change of variables gives

$$Z^*_{N+1}(\beta) = \int_{\mathfrak{o}^N} \prod_i |x_i|^{s_i} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}$$

  and hence $\mathbb{E}[\prod_i |X_i|^{s_i}] = \frac{Z^*_{N+1}(\beta)}{Z_N(\beta)}$.

- *Potential energy:* $\frac{d}{d\beta}$ can pass through the integral for Z_N, so

$$\mathbb{E}[V(\vec{X})] = -\frac{d}{d\beta} \log Z_N(\beta) \; .$$

Expectation of max's and min's

- For $\alpha_1, \alpha_2 \geq 0$ and $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, define

$$f(\vec{\alpha}, \vec{n}, \vec{x}) := (\max_{i<j} v(x_i - x_j))^{n_1} (\min_{i<j} v(x_i - x_j))^{n_2} (\min_{i<j} |x_i - x_j|)^{\alpha_1} (\max_{i<j} |x_i - x_j|)^{\alpha_2},$$

- *An important auxiliary function:* Define

$$F_N(\vec{\alpha}, \vec{\beta}) := \int_{(\mathfrak{m})^N} (\min_{i<j} |x_i - x_j|)^{\alpha_1} (\max_{i<j} |x_i - x_j|)^{\alpha_2} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}$$

for all suitable $(\vec{\alpha}, \vec{\beta}) \in \mathbb{C}^2 \times \mathbb{C}^{\binom{N}{2}}$.

- If $\beta > 0$ and $\beta_{ij} = q_i q_j \beta$ as before, note

$$\mathbb{E}[f(\vec{\alpha}, \vec{n}, \vec{X})] = \left(\frac{-1}{\log(q)} \frac{\partial}{\partial \alpha_1} \right)^{n_1} \left(\frac{-1}{\log(q)} \frac{\partial}{\partial \alpha_2} \right)^{n_2} \left[q^{\alpha_1 + \alpha_2} \cdot \frac{F_N(\vec{\alpha}, \vec{\beta})}{F_N(0, \vec{\beta})} \right] .$$

More about F_N

- If $\vec{\beta} = (\beta_{ij})_{1 \le i < j \le N}$ with $\beta_{ij} = q_i q_j \beta$ and $\vec{\beta}' = (\beta_{ij})_{1 \le i < j \le N+1}$ with $\beta_{i(N+1)} = s_i$ (where $\beta > 0$ and $\Re(s_i) > -1$), then

$$Z_N(\beta) = q^{N + \sum_{1 \le i < j \le N} \beta_{ij}} \cdot F_N(0, \vec{\beta}) \ ,$$
$$Z^*_{N+1}(\beta) = q^{N+1+\sum_{1 \le i < j \le N+1} \beta_{ij}} \cdot F_{N+1}(0, \vec{\beta}') \ ,$$

  and hence $\mathbb{E}[\prod_i |X_i|^{s_i}] = q^{1+\sum_i s_i} \cdot \frac{F_{N+1}(0,\vec{\beta}')}{F_N(0,\vec{\beta})}$.

- *Big idea:* The canonical partition function $Z_N(\beta)$ and the expectations $\mathbb{E}[\prod_i |X_i|^{s_i}]$, $\mathbb{E}[V(\vec{X})]$, and $\mathbb{E}[f(\vec{\alpha}, \vec{n}, \vec{X})]$ can all be expressed in terms of F_N.

- We will show that F_N can be computed via combinatorics.

A combinatorial gadget

Definition

Let L be a positive integer. A **splitting sequence of N** is a tuple $\vec{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{L-1})$ of compositions (i.e., ordered partitions) $\lambda_\ell = [\lambda_\ell^{(1)}, \lambda_\ell^{(2)}, \ldots, \lambda_\ell^{(N_\ell)}] \vdash N_{\ell+1}$ such that

$$1 = N_0 < N_1 < N_2 < \cdots < N_L = N.$$

Call $L(\vec{\lambda}) := L$ the **length** of $\vec{\lambda}$ and denote the set of all splitting sequences of N by $\pitchfork(N)$.

Example

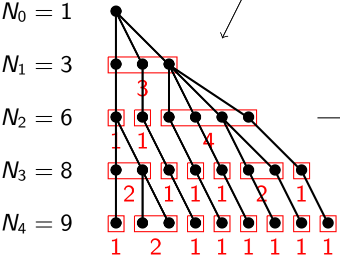
The tuple $\vec{\lambda} = ([3],\ [1,1,4],\ [2,1,1,1,2,1],\ [1,2,1,1,1,1,1,1])$ is a length 4 splitting sequence of 9.

Other ways to think about a splitting sequence

Our definition of splitting sequences has been chosen for brevity. There are two equivalent and useful alternatives.

Example

$$\vec{\lambda} = ([3], [1, 1, 4], [2, 1, 1, 1, 2, 1], [1, 2, 1, 1, 1, 1, 1, 1])$$



$\Lambda_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\Lambda_1 = \{1, 2, 3\}\{4\}\{5, 6, 7, 8, 9\}$

$\Lambda_2 = \{1, 2, 3\}\{4\}\{5\}\{6\}\{7, 8\}\{9\}$

$\Lambda_3 = \{1\}\{2, 3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}$

Special symbols associated to a splitting sequence

- If $\vec{\lambda} \in \pitchfork(N)$ and $\Lambda_0, \Lambda_1, \ldots, \Lambda_{L(\vec{\lambda})-1}$ are the corresponding partitions of $\{1, 2, \ldots, N\}$, denote the mth part of Λ_ℓ by $\Lambda_\ell^{(m)}$.

Definition

For each $\vec{\lambda} \in \pitchfork(N)$ and $\ell \in \{0, 1, \ldots, L(\vec{\lambda}) - 1\}$, define the ℓ**th multiplicity** and ℓ**th exponent** respectively by

$$M_\ell(\vec{\lambda}, n) := \prod_{m=1}^{N_\ell} \frac{1}{n} \binom{n}{\lambda_\ell^{(m)}} \qquad \text{and}$$

$$E_\ell(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) := \alpha_1 + \delta_{0\ell}\alpha_2 + \sum_{m=1}^{N_\ell} \left(|\Lambda_\ell^{(m)}| - 1 + \sum_{\substack{i<j \\ i,j \in \Lambda_\ell^{(m)}}} \beta_{ij} \right) .$$

$\vec{\lambda} = ([3],\ [1,1,4],\ [2,1,1,1,2,1],\ [1,2,1,1,1,1,1,1])$

- $\Lambda_0 = \{1,2,3,4,5,6,7,8,9\}$

$$M_0(\vec{\lambda}, n) = \frac{1}{n}\binom{n}{3}, \quad E_0(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + \alpha_2 + 8 + \sum_{1 \le i < j \le 9} \beta_{ij}$$

- $\Lambda_1 = \{1,2,3\}\{4\}\{5,6,7,8,9\}$

$$M_1(\vec{\lambda}, n) = \frac{1}{n}\binom{n}{4}, \quad E_1(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + 6 + \sum_{\substack{1 \le i < j \le 3 \\ 5 \le i < j \le 9}} \beta_{ij}$$

- $\Lambda_2 = \{1,2,3\}\{4\}\{5\}\{6\}\{7,8\}\{9\}$

$$M_2(\vec{\lambda}, n) = \left[\frac{1}{n}\binom{n}{2}\right]^2, \quad E_2(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + 3 + \beta_{12} + \beta_{13} + \beta_{23} + \beta_{78}$$

- $\Lambda_3 = \{1\}\{2,3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}$

$$M_3(\vec{\lambda}, n) = \frac{1}{n}\binom{n}{2}, \quad E_3(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + 1 + \beta_{23}$$

An explicit formula for F_N

- Define $\sigma(\vec{\beta}) := (\beta_{\sigma^{-1}(i)\sigma^{-1}(j)})$ if $\sigma \in S_N$ and define open sets

$$\Omega^+ := \{(\vec{\alpha},\vec{\beta}) : \Re(E_\ell(\vec{\alpha},\sigma(\vec{\beta}),\vec{\lambda})) > 0 \text{ for all } \sigma,\vec{\lambda},\ell\} \ ,$$
$$\Omega := \{(\vec{\alpha},\vec{\beta}) : E_\ell(\vec{\alpha},\sigma(\vec{\beta}),\vec{\lambda}) \notin \tfrac{2\pi i\mathbb{Z}}{\log(q)} \text{ for all } \sigma,\vec{\lambda},\ell\} \ .$$

Theorem (W.)

The function F_N defined by

$$F_N(\vec{\alpha},\vec{\beta}) := \int_{(\mathfrak{m})^N} (\min_{i<j}|x_i-x_j|)^{\alpha_1}(\max_{i<j}|x_i-x_j|)^{\alpha_2}\prod_{i<j}|x_i-x_j|^{\beta_{ij}}\,d\vec{x}$$

is analytic on Ω^+ and extends to all of Ω via

$$F_N(\vec{\alpha},\vec{\beta}) = \frac{1}{q^N}\sum_{\sigma\in S_N}\sum_{\vec{\lambda}\in\pitchfork(N)}\prod_{\ell=0}^{L(\vec{\lambda})-1}\frac{M_\ell(\vec{\lambda},q)}{q^{E_\ell(\vec{\alpha},\sigma(\vec{\beta}),\vec{\lambda})}-1} \ .$$

A fun corollary

- For all $\sigma \in S_N$ and all $\vec{\lambda} \in \pitchfork(N)$ we have

$$E_0(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda}) = \alpha_1 + \alpha_2 + N - 1 + \sum_{i<j} \beta_{ij}$$

  so $(q^{E_0(\vec{\alpha},\sigma(\vec{\beta}),\vec{\lambda})} - 1)^{-1}$ is independent of σ and $\vec{\lambda}$ and factors out of the whole sum.

- If $\ell > 0$, $E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda})$ is independent of α_2.

Corollary

If $\alpha \geq 0$, $\beta > 0$, and $\beta_{ij} = q_i q_j \beta$, then

$$\mathbb{E}[(\max_{i<j} |X_i - X_j|)^\alpha] = \frac{q^{N-1+\sum_{i<j} \beta_{ij}} - 1}{q^{N-1+\sum_{i<j} \beta_{ij}} - q^{-\alpha}}.$$

Another fun corollary

- If $\alpha_1 = \alpha_2 = 0$ and all $q_i = 1$, then $\beta_{ij} = \beta$ for all ij, then

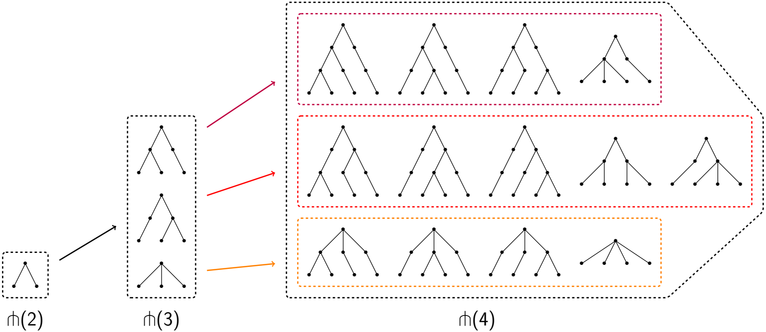
$$E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda}) = E_\ell(0, \vec{\beta}, \vec{\lambda}) = \sum_{m=1}^{N_\ell} \left(|\Lambda_\ell^{(m)}| - 1 + \binom{|\Lambda_\ell^{(m)}|}{2} \beta \right)$$

for all $\sigma \in S_N$ and we get an analogue of Mehta's integral:

Corollary

$$\int_{K^N} \mathbf{1}_{\mathfrak{o}^N}(\vec{x}) \prod_{i<j} |x_i - x_j|^\beta \, d\vec{x} = N! q^{\binom{N}{2}\beta} \sum_{\vec{\lambda} \in \pitchfork(N)} \prod_{\ell=0}^{L(\vec{\lambda})-1} \frac{M_\ell(\vec{\lambda}, q)}{q^{E_\ell(0,\vec{\beta},\vec{\lambda})} - 1} .$$

Recursive construction of $\pitchfork(N)$



All splitting sequences in $\pitchfork(N)$ can be constructed by adding nodes and edges to those in $\pitchfork(N-1)$. An inductive argument gives $|\pitchfork(N)| \leq (2N-3)!!$ with equality only if $N = 2$ or $N = 3$.

$Z_N(\beta)$ values revisited

- Everything simplifies considerably when all $q_i = 1$:
  - $Z_2(\beta) = \frac{(q-1)q^{\beta}}{q^{\beta+1}-1}$
  - $Z_3(\beta) = \frac{(q-1)q^{3\beta}}{q^{3\beta+2}-1} \cdot \left[(q-2) + \frac{3(q-1)}{q^{\beta+1}-1}\right]$
  - $$Z_4(\beta) = \frac{(q-1)q^{6\beta}}{q^{6\beta+3}-1} \cdot \left\{(q-2)(q-3) + \frac{6(q-1)(q-2)}{q^{\beta+1}-1} + \frac{4(q-1)(q-2)}{q^{3\beta+2}-1} + \frac{12(q-1)^2}{(q^{3\beta+2}-1)(q^{\beta+1}-1)} + \frac{3(q-1)^2}{(q^{\beta+1}-1)^2}\right\}$$

Thank you!