The Fourier series of Bernoulli polynomials

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The mth *Bernoulli number* is defined to be the coefficient of $z^m/m!$ in the Taylor expansion for $\frac{z}{e^z-1}$. That is,

$$\frac{z}{e^z-1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m,$$

which converges absolutely when $|z| < 2\pi$. It is straightforward to verify that the even and odd parts of $\frac{z}{e^z-1}$ are respectively $\frac{z}{2} \cdot \frac{e^z+1}{e^z-1}$ and $-\frac{z}{2}$, and therefore

$$\frac{z}{2} \cdot \frac{e^z+1}{e^z-1} = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k} \qquad \text{and} \qquad -\frac{z}{2} = \sum_{k=0}^{\infty} \frac{B_{2k+1}}{(2k+1)!} z^{2k+1}$$

when $|z| < 2\pi$. The righthand equation shows that $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$ for all $k \geq 1$, and evaluating the lefthand equation at $z = 2\pi i w$ shows that

$$\pi w \cot(\pi w) = \sum_{k=0}^{\infty} \frac{(2\pi)^{2k}(-1)^k B_{2k}}{(2k)!} w^{2k} \qquad \text{when} \quad |w| < 1. \tag{0.0.1}$$

A formula for $\zeta(2\ell) = \sum_{n=1}^{\infty} \frac{1}{n^{2\ell}}$ can be found for any positive integer ℓ by expanding $\pi w \cot(\pi w)$ as a sum over its poles and recombining it into a power series in w. The main goal of this note is to establish the same formula (and a little more) using Fourier series instead. To this end, we define a sequence of 1-periodic functions $P_m : \mathbb{R} \to \mathbb{R}$ via $P_1(x) := \{x\} - \frac{1}{2}$ (where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x) and

$$P_m(x) = B_m + m \int_0^x P_{m-1}(t)\, dt \qquad \text{for all } m \geq 2.$$

Obviously $P_m(0) = B_m$ for all $m \geq 1$, and it is a good exercise to verify that P_m is indeed 1-periodic, smooth on $\mathbb{R} \setminus \mathbb{Z}$ for $m \geq 1$, and continuous on $\mathbb{R}$ if $m \geq 2$. In fact, for every $m \geq 1$, $P_m(x)$ is a polynomial in $\{x\}$ (but not in x), and is thus called the mth *periodic Bernoulli polynomial.*

The Basel Problem

The Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ for the 1-periodic function $P_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}$ is given by $c_0 = \int_0^1 P_2(x)\, dx = 0$ and

$$c_n = \int_0^1 P_2(x) e^{-2\pi i n x}\, dx = \frac{1}{2\pi^2 n^2} = -2 \cdot \frac{1}{(2\pi i n)^2} \qquad \text{for all } n \neq 1.$$

Since P_2 is continuous and piecewise smooth, a theorem of Dirichlet implies that the Fourier series converges uniformly to P_2 on $\mathbb{R}$. In particular,

$$P_2(x) = -2\sum_{n=1}^{\infty} \frac{e^{2\pi inx} + e^{-2\pi inx}}{(2\pi in)^2} \qquad \text{for all } x \in \mathbb{R}. \tag{0.0.2}$$

The solution to the Basel problem (i.e., $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$) follows immediately by evaluating both sides of (0.0.2) at $x = 0$.

Formulas for $P_m(x)$ and $\zeta(4), \zeta(6), \zeta(8)$...

The recursive definition of P_m, the operator $I_{[0,x]}$ defined by $I_{[0,x]}(f) = \int_0^x f(t)\,dt$, and a straightforward induction argument give the formula

$$P_m(x) = \sum_{k=0}^{m-3} \binom{m}{k} B_{m-k} x^k + \frac{m!}{2!} \cdot I_{[0,x]}^{m-2}(P_2) \qquad \text{for all } m \geq 2, \tag{0.0.3}$$

where $I_{[0,x]}^{m-2}$ stands for $(m-2)$th iterate of the operator $I_{[0,x]}$. It can also be applied to the uniformly convergent sum for P_2 in (0.0.2) term-by-term as many times as we like:

$$\begin{aligned}
I_{[0,x]}^{3-2}(P_2) &= -2\sum_{n=1}^{\infty} \frac{e^{2\pi inx} - e^{-2\pi inx}}{(2\pi in)^3} \\
I_{[0,x]}^{4-2}(P_2) &= -2\sum_{n=1}^{\infty} \frac{e^{2\pi inx} + e^{-2\pi inx}}{(2\pi in)^4} + 4 \cdot \frac{\zeta(4)}{(2\pi i)^4} \\
I_{[0,x]}^{5-2}(P_2) &= -2\sum_{n=1}^{\infty} \frac{e^{2\pi inx} - e^{-2\pi inx}}{(2\pi in)^5} + 4 \cdot \frac{\zeta(4)}{(2\pi i)^4} x \\
I_{[0,x]}^{6-2}(P_2) &= -2\sum_{n=1}^{\infty} \frac{e^{2\pi inx} + e^{-2\pi inx}}{(2\pi in)^6} + 4\left(\frac{\zeta(4)}{(2\pi i)^4}\frac{x^2}{2!} + \frac{\zeta(6)}{(2\pi i)^6}\right) \\
&\vdots
\end{aligned}$$

By keeping track of the two sums that emerge (or more carefully, by induction), we conclude that

$$\frac{m!}{2!} \cdot I_{[0,x]}^{m-2}(P_2) = \sum_{\ell=2}^{\lfloor m/2 \rfloor} \binom{m}{2\ell} \frac{2(2\ell)!}{(2\pi i)^{2\ell}} \zeta(2\ell) x^{m-2\ell} - \sum_{n=1}^{\infty} \frac{m!}{(2\pi in)^m} \left(e^{2\pi inx} + (-1)^m e^{-2\pi inx}\right) .$$

On the other hand, we know that $B_{2\ell+1} = 0$ for all $\ell \geq 1$, and thus

$$\sum_{k=0}^{m-3} \binom{m}{k} B_{m-k} x^k = \sum_{k=3}^{m} \binom{m}{k} B_k x^{m-k} = \sum_{\ell=2}^{\lfloor m/2 \rfloor} \binom{m}{2\ell} B_{2\ell} x^{m-2\ell} ,$$

so we can regroup (0.0.3) as

$$\begin{aligned}
P_m(x) &= \sum_{\ell=2}^{\lfloor m/2 \rfloor} \binom{m}{2\ell} \left[B_{2\ell} + \frac{2(2\ell)!}{(2\pi i)^{2\ell}}\zeta(2\ell)\right] x^{m-2\ell} - \sum_{n=1}^{\infty} \frac{m!}{(2\pi in)^m} \left(e^{2\pi inx} + (-1)^m e^{-2\pi inx}\right) \\
&= \sum_{\ell=2}^{\lfloor m/2 \rfloor} \binom{m}{2\ell} \left[B_{2\ell} + \frac{2(2\ell)!}{(2\pi i)^{2\ell}}\zeta(2\ell)\right] x^{m-2\ell} - \sum_{n=1}^{\infty} \frac{m!}{(2\pi n)^m} \left(e^{2\pi inx - \frac{\pi i}{2}m} + e^{-(2\pi inx - \frac{\pi i}{2}m)}\right) \\
&= \sum_{\ell=2}^{\lfloor m/2 \rfloor} \binom{m}{2\ell} \left[B_{2\ell} + \frac{2(2\ell)!}{(2\pi i)^{2\ell}}\zeta(2\ell)\right] x^{m-2\ell} - \sum_{n=1}^{\infty} \frac{2 \cdot m!}{(2\pi n)^m} \cos\left(2\pi nx - \frac{\pi}{2}m\right).
\end{aligned}$$

If m is odd and greater than 4, we can see that the polynomial $\sum_{\ell=2}^{\lfloor m/2 \rfloor} \binom{m}{2\ell} \left[B_{2\ell} + \frac{2(2\ell)!}{(2\pi i)^{2\ell}} \zeta(2\ell)\right] x^{m-2\ell}$ has positive degree and no constant term, but $P_m(x)$ is bounded on $\mathbb{R}$ (because it is continuous and periodic) and the rightmost sum of cosines is also bounded on $\mathbb{R}$ (because it is uniformly convergent), so it must be the case that $B_{2\ell} + \frac{2(2\ell)!}{(2\pi i)^{2\ell}} \zeta(2\ell) = 0$ for all $\ell \in \{2, 3, \ldots, \lfloor m/2 \rfloor\}$. Since m can be arbitrarily large and since we already solved the $\ell = 1$ case (the Basel problem), we conclude with the following theorem:

Theorem 0.1.

(a) For any integer $\ell \geq 1$, the sum $\zeta(2\ell) = \sum_{n=1}^{\infty} \frac{1}{n^{2\ell}}$ is given by

$$\zeta(2\ell) = -\frac{(2\pi i)^{2\ell} B_{2\ell}}{2(2\ell)!} = \frac{(2\pi)^{2\ell} |B_{2\ell}|}{2(2\ell)!}.$$

(b) For $m \geq 2$, the periodic Bernoulli polynomial can be expressed as an absolutely uniformly convergent series:

$$P_m(x) = -\frac{2 \cdot m!}{(2\pi)^m} \sum_{n=1}^{\infty} \frac{1}{n^m} \cos\left(2\pi n x - \frac{\pi}{2} m\right).$$

(To play with some visual examples, check out this Desmos example.)